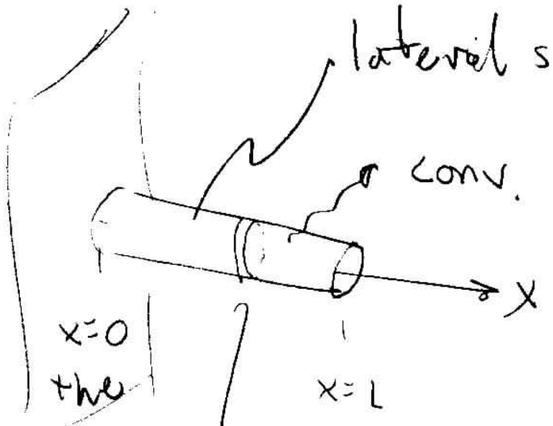
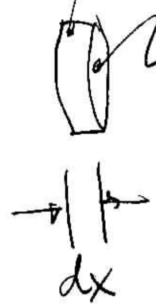
Heat Transfer from Extended Surfaces



lateral surface and x-sec. area vary with x

conv. heat transfer out to surrounding fluid at T_∞, with h

slice from "pin"

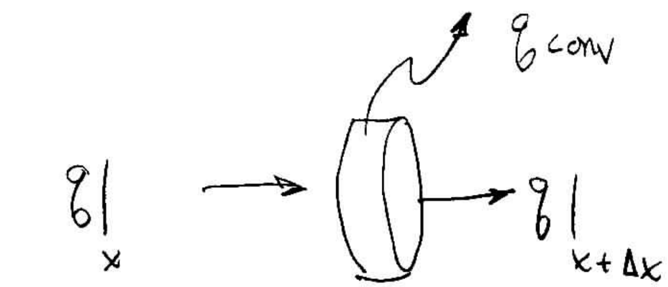


perimiter

area $S = P \, dx$

area A_c of cross section

Cons. of energy



$$q|_x = q|_{x+\Delta x} + q_{conv}$$

$$q|_{x+\Delta x} - q|_x = -q_{conv}$$

$$\left(\frac{dq}{dx}\right)\Delta x = -q_{conv}$$

But recall $q = -k \frac{dT}{dx} A_c$ so

$$\left(\frac{dq}{dx}\right) \Delta x = -q_{conv}$$

$$\frac{d}{dx}\left(-k \frac{dT}{dx} A\right) dx = -hS\left(T_{surf} - T_\infty\right)$$

$$-k \frac{d}{dx}\left(A_c \frac{dT}{dx}\right) dx = -hS\left(T(x) - T_\infty\right)$$

- Assumes entire pin cross-section is at the same temperature (but it can varry in the x-dir.)
- Assumes $k = ¢$

$$\frac{d}{dx}\left(A_c(x) \frac{dT}{dx}\right) dx = \frac{h S(x)}{k}\left(T - T_\infty\right)$$

If $A_c(x) = ¢ = A_c$ then finally

$$\frac{d^2T}{dx^2} dx = \frac{hS}{kA_c}\left(T - T_\infty\right) = \frac{h(P \cdot dx)}{kA_c}\left(T - T_\infty\right)$$

or

$$\frac{d^2T}{dx^2} = \frac{hP}{kA_c}\left(T - T_\infty\right)$$

If we define $\Theta(x) = T(x) - T_\infty$

$$\boxed{\frac{d^2\Theta}{dx^2} = \left(\frac{hP}{kA_c}\right)\Theta = m^2\Theta} \quad \text{where} \quad \boxed{m^2 = \frac{hP}{kA_c}}$$

This has solutions $\cosh(mx)$ and $\sinh(mx)$

so

$$\Theta = A\cosh(mx) + B\sinh(mx)$$

we know that at $\boxed{x=0\ ,\ T=T_b}$ B.C. #1

so

$\Theta_b = T_b - T_\infty = A$ so $\Theta = \Theta_b \cosh(mx) + B\sinh(mx)$

Hint: we need one more B.C. #2

Several options

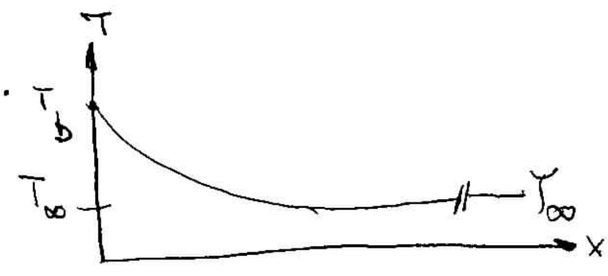
1) $\underline{\infty}$ $\underline{\text{long}}$ $\underline{\text{fin}}$ then $\Theta\Big|_{x=L\to\infty} = T\Big|_{L\to\infty} - T_\infty = 0$

in which case

$$\frac{\Theta}{\Theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}}$$

and $q_{fin} = -kA_c \frac{dT}{dx}\Big|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$

because all heat lost from the pin surface had to come from (through) the base.



2) Adiabatic Tip (insulated or negligible loss from tip)

$$\dot{q}\Big|_{x=tip} = 0 \Rightarrow \frac{d\theta}{dx}\Big|_{x=L} = 0$$

leads to

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

so then $q_{fin} = -kA_c \frac{dT}{dx}\Big|_{x=0} = \sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL$

Note that as $L \to \infty$, $\tanh mL \to 1$ and so

$\lim_{L\to\infty}$ (insulated tip) $\to$ (∞-long fin)

3) Fixed Tip temp $\quad \theta\Big|_{x=L} = \theta_L = T_L - T_\infty$

T_L ← specified value

$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = \left(\frac{T_L - T_\infty}{T_b - T_\infty}\right)\frac{\sinh mx + \sinh m(L-x)}{\sinh mL}$$

in which case

$$q_{fin} = -kA_c \frac{dT}{dx}\Big|_{x=0} = \sqrt{hPkA_c}\,(T_b - T_\infty)\,\frac{\cosh mL - \left(\frac{T_L - T_\infty}{T_b - T_\infty}\right)}{\sinh mL}$$

Note that as $L \to \infty$ these also reduce to the ∞-long pin results.

4) Active Tip (Convection from tip)

B.C. is now $-kA_c \left.\frac{dT}{dx}\right|_{x=L} = hA_c \left(T|_{x=L} - T_\infty\right)$

Lots of fun algebra . . .

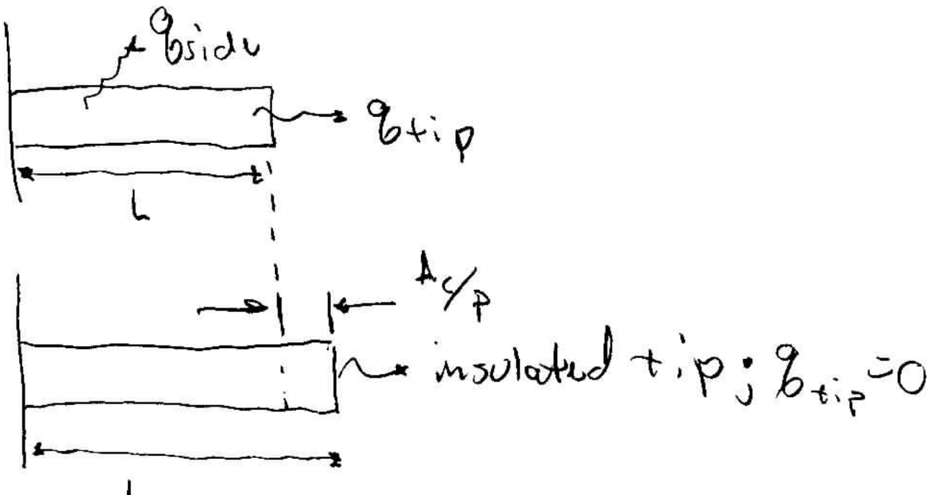
$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + \left(\frac{h}{mk}\right)\sinh m(L-x)}{\cosh mL + \left(\frac{h}{mk}\right)\sinh mL}$$

and so

$$q_{fin} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hPkA_c}\,(T_b - T_\infty)\,\frac{\sinh mL + \left(\frac{h}{mk}\right)\cosh mL}{\cosh mL + \left(\frac{h}{mk}\right)\sinh mL}$$

Here's an idea . . .

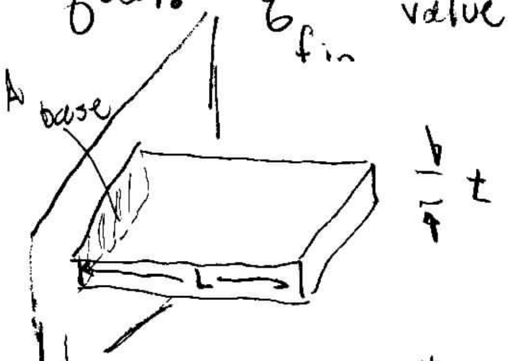
Use the insulated tip solution but add a bit of length to the fin to get the equal. q_{fin} value . . .



$$\boxed{L_{cor} = L + A_c/P}$$ with insulated tip

good for $mL \geq 1$

$\boxed{\text{active tip}}$

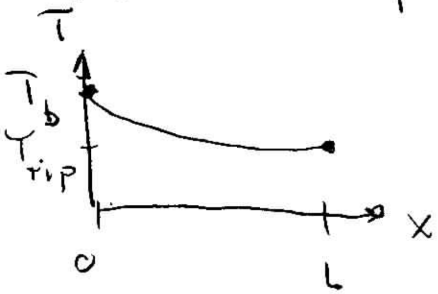


L_{cor} rectangular fin $= L + \frac{t}{2}$

L_{cor} cylindrical fin $= L + \frac{D}{4}$

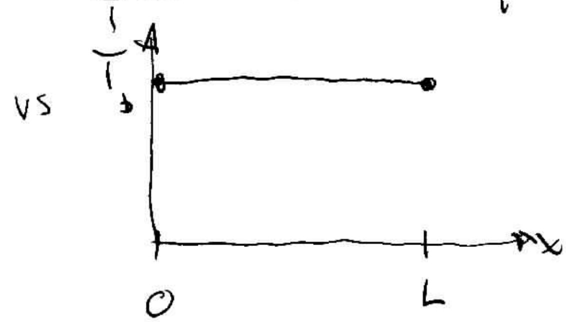
Fin efficiency

Actual fin temp



vs

"Ideal" fin temp



(Really good k)

Define fin efficiency $\equiv \dfrac{\dot{q}_{fin,\,actual}}{\dot{q}_{fin,\,ideal}} = \eta_{fin}$

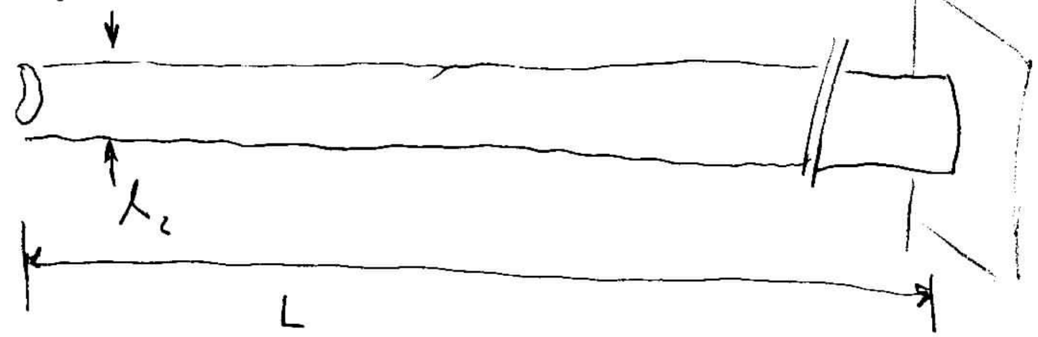
$$\eta_{\substack{adiabatic\\tip}} = \frac{\dot{q}_{fin}}{\dot{q}_{fin\ max}} = \frac{\sqrt{hPkA_c}\,(T_b - T_\infty)\tanh mL}{hA_{fin}(T_b - T_\infty)} = \frac{\tanh mL}{mL}$$

$A_{fin} = PL$

$$\eta_{\substack{long\\fin}} = \frac{\sqrt{hPkA}\,(T_b - T_\infty)}{hA_{fin}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hP}} = 1/mL$$

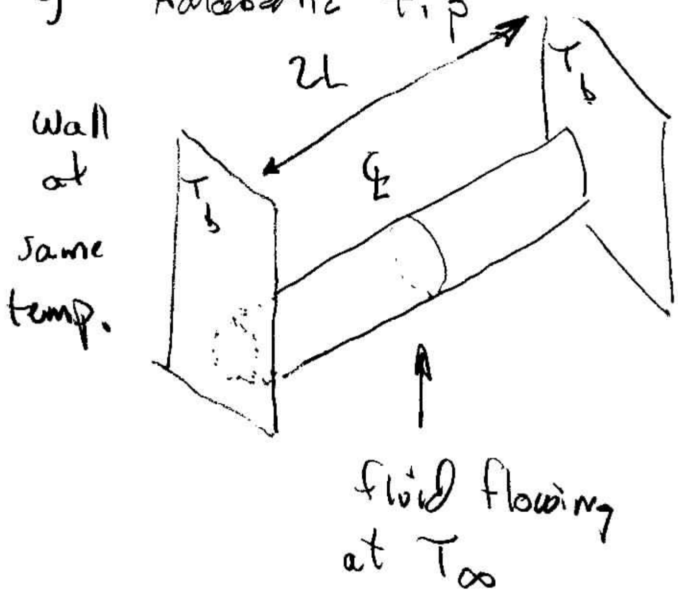
So how does one physically create the conditions for each of the 4 cases?

1) ∞-long — just make really long compared to the characteristic dimension of its cross section

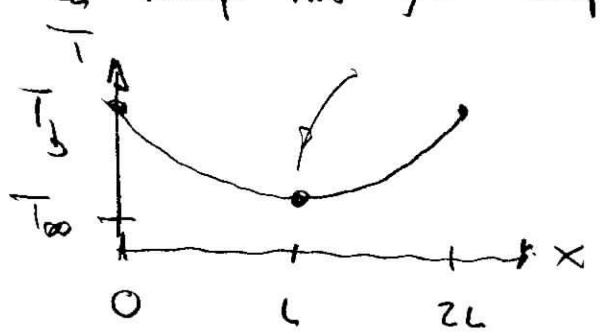


keep $\frac{l_c}{L} << 1$

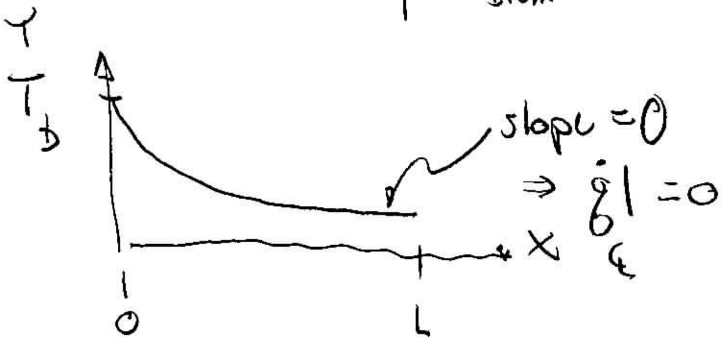
2) Adiabatic tip



℄ temp has zero slope

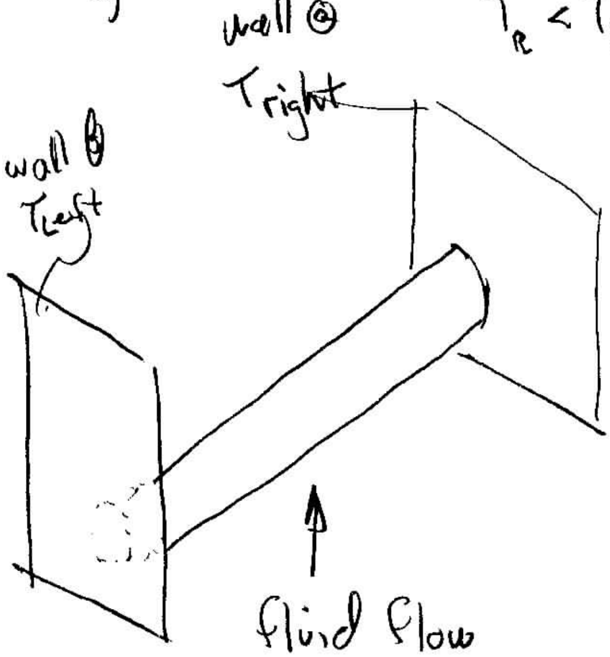


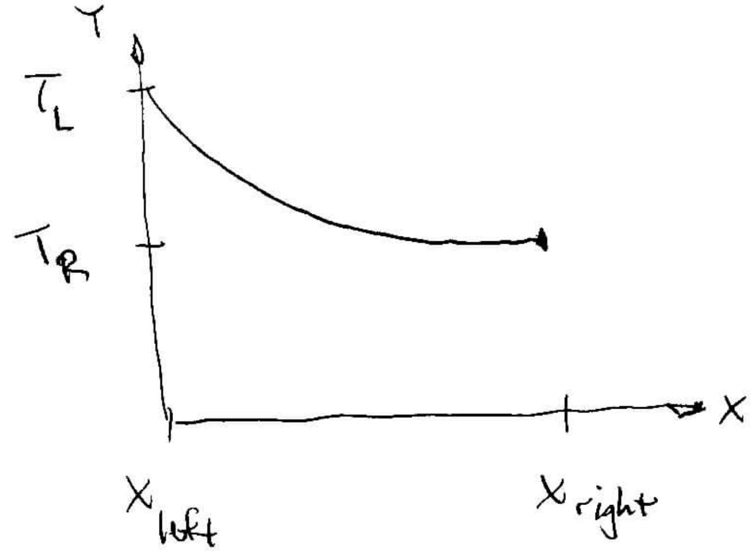
so just analyze $\frac{1}{2}$ of the problem



3) Fixed Tip Temp

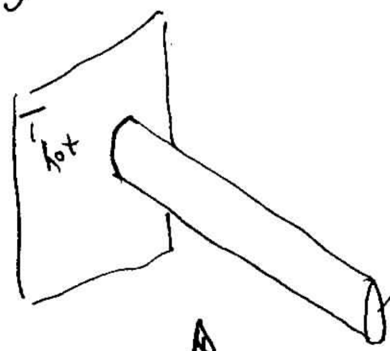
$T_R < T_L$





fluid flow
@ T_∞

4) Active tip



comp. surface area relative to lateral surface area

fluid flow
@ T_∞
including over the tip

Fin effectiveness

$$\varepsilon_{fin} \equiv \frac{q_{fin}}{q_{no\,fin}} = \frac{q_{fin}}{hA_b(T_b-T_\infty)} = \frac{q \text{ transfer from just } A_b}{q \text{ transfer with fin of } A_b}$$

if $\varepsilon = 1$ no point to adding the fin

$\varepsilon < 1$ fin acts as an insulator (more is not always better)

$\varepsilon > 1$ fin does its job ~ removes more heat than if it were not there

Note that ε and η are related...

$$\varepsilon_{fin} = \frac{q_{fin}}{q_{no\,fin}} = \frac{q_{fin}}{hA_b(T_b-T_\infty)}$$

$$= \frac{\eta_{fin}\, h A_{fin} (T_b-T_\infty)}{h A_b (T_b-T_\infty)} = \frac{A_{fin}}{A_b}\eta_{fin}$$

Special case - long fin
- uniform cross section $(A_b = A_c)$
- steady state

then $$\varepsilon_{\substack{long\\fin}} = \frac{q_{fin}}{q_{no\,fin}} = \frac{\sqrt{hPkA_c}\,(T_b-T_\infty)}{hA_b(T_b-T_\infty)} = \sqrt{\frac{kP}{hA_c}}$$

General guidelines for fin design

- use $k\uparrow$ materials - metals
  - copper good, but \$\$\$
  - Aluminum good compromise, \$
  - Iron so,so \$

- use $\frac{P}{A_c}\uparrow$ as possible - thin plates (guts of computers), slender pins

- use with low convection ($\downarrow h$) environments (computers)
  - gas (liquids are better!)
  - natural convection (use fans or car radiators)

Overall ε

$q_{total} = q_{no\,fin} + q_{fin}$ (q from whole surface)

$$= h A_{no\,fin}(T_b - T_\infty) + \eta_{fin} h A_{fin}(T_b - T_\infty)$$

$$= h\left(A_{no\,fin} + \eta_{fin} A_{fin}\right)(T_b - T_\infty)$$

lots of fins but also blank areas

so

$$\varepsilon_{fin\,overall} = \frac{q_{total,fin}}{q_{total,no\,fin}} = \frac{h\left(A_{no\,fin} + \eta_{fin} A_{fin}\right)(T_b - T_\infty)}{h A_{no\,fin\,at\,all}(T_b - T_\infty)}$$

$$= \frac{A_{no\,fin} + \eta_{fin} A_{fin}}{A_{no\,fin\,at\,all}}$$